Poynting theorem: Law of conservation of energy

For a charge q, rate of doing work by external electromagnetic fields $\vec{E}$ and $\vec{B}$ is $= q\vec{E}\cdot\frac{dx}{dt} = q\vec{E}\cdot\vec{v} = \int(\rho\, dV)\,\vec{v}\cdot\vec{E}$

$$= \int_V \vec{J}\cdot\vec{E}\, dV \qquad \because \rho\vec{v}=\vec{J}$$

Conversion of e.m. energy into mechanical energy

$$\int_V \vec{J}\cdot\vec{E}\, dV = \int_V \vec{E}\cdot\left(\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right) dV$$

$$= \int_V \left[\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] dV$$

$$= \int_V \left[\vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] dV$$

$$= -\int_V \left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] dV \quad \text{— (1)}$$

$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$

Now energy density of e.m. field $u = \frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{H}\cdot\vec{B}$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2}\left[\frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \vec{D}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{H}}{\partial t}\cdot\vec{B} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]$$

$$= \frac{1}{2}\left[\epsilon\left(2\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right) + \frac{1}{\mu}\left(2\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right)\right] = \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$$

Eqn (1) $\Rightarrow$

$$\int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right] dV = -\int_V \vec{J}\cdot\vec{E}\, dV \qquad \vec{S} = \vec{E}\times\vec{H}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$ Differential form

$$\Rightarrow \int_V \frac{\partial u}{\partial t}\, dV = \int_V \frac{\partial}{\partial t}(u\, dV) = \frac{\partial}{\partial t}\left[\int_V u\, dV\right] = \frac{\partial}{\partial t} U_{field}$$

and $\quad \int \vec{J}\cdot\vec{E}\, dV = \frac{\partial}{\partial t} U_{mech.}$

$$\therefore \quad \frac{\partial}{\partial t} U_{field} + \frac{\partial}{\partial t} U_{mech.} = -\int_V \vec{\nabla}\cdot\vec{S}\, dV$$

$$\Rightarrow \boxed{\frac{\partial U}{\partial t} = -\oint_S \vec{S}\cdot d\vec{a}}$$

Conservation of energy for combined system of particles and fields.

Statement Rate of increase of energy in any volume in an electromagnetic field is equal to the rate of energy flow inside the surface enclosing the volume.

Conservation of linear momentum

e.m. Force on charge q is $\vec{F} = q\,(\vec{E} + \vec{v}\times\vec{B})$

Newton's law

$$\frac{\partial \vec{P}_{mech}}{\partial t} = \int_V [\rho\vec{E} + \vec{J}\times\vec{B}]\,dV$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon(\vec{\nabla}\cdot\vec{E})\vec{E} + \left[\vec{\nabla}\times\frac{\vec{B}}{\mu} - \epsilon\frac{\partial\vec{E}}{\partial t}\right]\times\vec{B} \Leftarrow \text{Maxwell eqns}$$

$$= \epsilon\left\{\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\times\frac{\partial\vec{E}}{\partial t}\right\} - \frac{1}{\mu}\left\{\vec{B}\times\vec{\nabla}\times\vec{B}\right\}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\therefore\ \vec{B}\times\frac{\partial\vec{E}}{\partial t} = \vec{E}\times\frac{\partial\vec{B}}{\partial t} - \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = -\vec{E}\times(\vec{\nabla}\times\vec{E}) - \mu\frac{\partial}{\partial t}\vec{S}$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon\left\{\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \mu\frac{\partial}{\partial t}\vec{S}\right\} + \frac{1}{\mu}\left\{\vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times\vec{\nabla}\times\vec{B}\right\}$$

$\Uparrow$ $\because\ \vec{B}(\vec{\nabla}\cdot\vec{B})$ as $\vec{\nabla}\cdot\vec{B} = 0$

$$\frac{\partial\vec{P}_{mech}}{\partial t} = \int_V \left[\epsilon\left\{\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right\} + \frac{1}{\mu}\left\{\vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right\}\right]dV - \int_V \mu\epsilon\frac{\partial}{\partial t}\vec{S}\,dV$$

$$\int_V \epsilon\mu\frac{\partial\vec{S}}{\partial t}\,dV = \frac{\partial}{\partial t}\left[\epsilon\mu\int_V \vec{S}\,dV\right] = \frac{\partial}{\partial t}\vec{P}_{field}$$

$$\Rightarrow \frac{\partial\vec{P}_{mech}}{\partial t} + \frac{\partial\vec{P}_{field}}{\partial t} = \frac{\partial\vec{P}}{\partial t} = \int_V \left[\epsilon\left\{\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right\} + \frac{1}{\mu}\left\{\vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right\}\right]dV$$

$$x_\alpha,\quad \alpha = 1, 2, 3\cdots$$

$\alpha = 1$ Component of electric part of this eqn is,

$$(\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}))_1 = E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) + E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}(E_1^2) + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3) - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 + E_2^2 + E_3^2)$$

$$\Rightarrow \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta - \frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right)$$

div. of 2nd rank tensor

Maxwell Stress Tensor, $T_{\alpha\beta} = \epsilon\left[E_\alpha E_\beta - \frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right] + \frac{1}{\mu}\left[B_\alpha B_\beta - \frac{1}{2}\vec{B}\cdot\vec{B}\,\delta_{\alpha\beta}\right]$

Then,

Momentum conservation

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}(P_{mech} + P_{field})_\alpha &= \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\,dV \\ &= \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da \end{aligned}}$$

n outward normal

$\sum_\beta T_{\alpha\beta} n_\beta$ is the α^{th} component of flow per unit area of momentum across surface S into the volume V.